M332 RB

THE OPEN UNIVERSITY
Mathematics: A Third Level Course

REVISION BOOKLET: COMPLEX ANALYSIS

Prepared by the Course Team

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R.0 INTRODUCTION

The purpose of this revision booklet is to help you to find out how well you have understood the various parts of the course. You should use it to diagnose your weaknesses, and should then refer back to those units which you feel the need to revise.

The booklet is in two parts. We start, in Section R.1, by summarizing the main topics covered in the first eleven units of the course, and giving you some Self-assessment Questions and Problems on these units. The Self-Assessment Questions are intended to be fairly straightforward, and should enable you to isolate those parts of the course which you most need to revise. The Problems, though harder, are intended to be of a routine nature.

We recommend that you work all the questions for a unit and then check the solutions (which follow the questions), before moving on to the next unit. If you can do most of these questions without looking up the relevant units, you can claim to have a good idea of the basic material of the course.

However, the questions in Section R.1, although important in testing your recollection of each separate unit, will not give you much opportunity to test your overall understanding of the course. We remedy this in Section R.2 where we have set some "open-ended" questions. These questions are designed to test your understanding of how the various units relate to each other, and how various results, introduced in one unit, are used in later units.

We do not expect you to try *every* question and problem in this revision booklet. There are many questions and problems here if you want them, but you may prefer to regard this booklet as something to dip into while preparing yourself for the examination.

R.1 SELF-ASSESSMENT QUESTIONS AND PROBLEMS

In this section we shall review the first eleven units one at a time. For each unit we shall remind you very briefly of the main topics covered in that unit, and you will then have a chance to see how much you remember by trying some questions. Some of these questions will be very similar to the Self-Assessment Questions given in the units, and are designed to help you diagnose those parts of the course which you most need to revise. Others will be somewhat harder, though still straightforward, and will be similar to the sort of problems you had in the units.

Unit 1 Complex Numbers

We began this unit by introducing the complex number system and showing that it is equivalent to $\mathbf{R} \times \mathbf{R}$ with suitably-defined operations of addition and multiplication. In particular, we showed how complex numbers can be represented as points in a plane, and we defined the real part ($\mathrm{Re}\, z$), imaginary part ($\mathrm{Im}\, z$), conjugate ($\bar{z}$), modulus ($|z|$), and the principal value of the argument ($\mathrm{Arg}\, z$), of a complex number z. We also showed how complex numbers can be written in polar form ($z = r(\cos\theta + i\sin\theta)$). We concluded with a proof of the Nested Rectangles Theorem.

1.1 For each of the following complex numbers z, evaluate

(a) $\mathrm{Re}\, z$, (b) $\mathrm{Im}\, z$, (c) $\bar{z}$, (d) $|z|$, (e) $\mathrm{Arg}\, z$.

(i) $z = 4i$ (ii) $z = 1 + i\sqrt{3}$ (iii) $z = (1 - i)^3$

1.2 Write down polar forms of the complex numbers in Question 1.1, and hence find their square roots. (You may, of course, write polar forms in the form $re^{i\theta}$.)

1.3 Draw diagrams to illustrate the following sets of points in the complex plane.

(i) $\{z: \mathrm{Re}\, z > -1\}$

(ii) $\{z: |\mathrm{Im}\, z| > 1\}$

(iii) $\{z: 1 < \mathrm{Re}\, z < 2 \text{ and } 1 \leq \mathrm{Im}\, z \leq 3\}$

(iv) $\{z: \mathrm{Re}\,(z^2) = 0\}$

(v) $\{z: |z| \geq 1, |\mathrm{Arg}\, z| \leq \frac{1}{4}\pi\}$

1.4 If z and w are complex numbers, classify the following statements as necessarily true, necessarily false, or sometimes true and sometimes false.

(i) $|\mathrm{Im}\, z| \leq |z|$

(ii) $2\,|\mathrm{Re}\, z| \cdot |\mathrm{Im}\, z| \leq |z|^2$

(iii) $|z + w|^2 - |z^2| - |w^2| = z\bar{w} + \bar{z}w$

(iv) $|z + w|^2 \leq |z|^2 + |w|^2$

(v) $|z - w| \leq \big||z| - |w|\big|$

1.5 Solve the equation $(z + 1)^7 = (z - 1)^7$. (Hint: Do not multiply out the brackets.)

Solutions

1.1

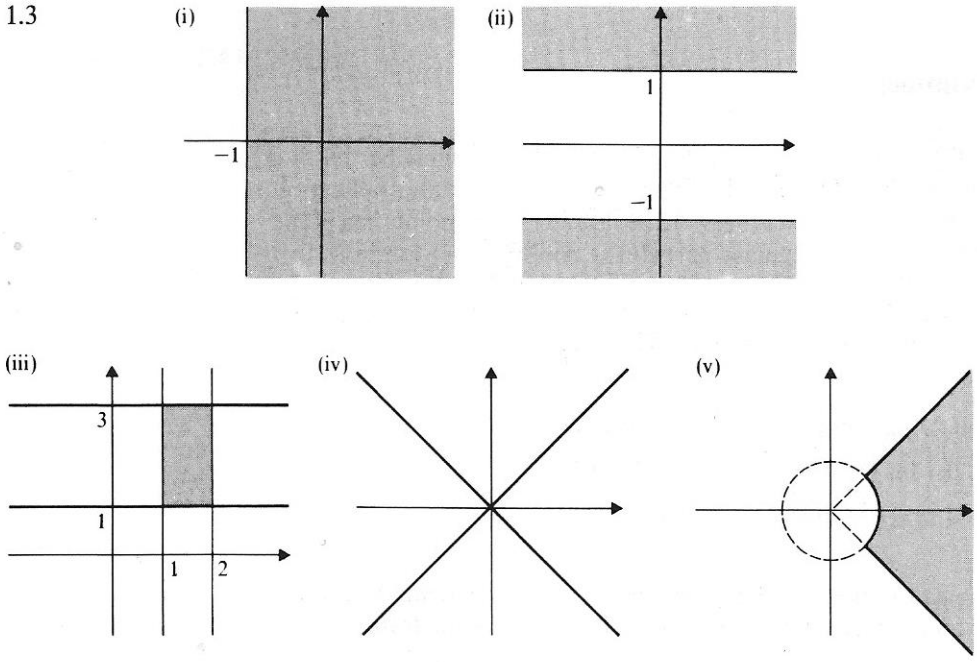
z	$\mathrm{Re}\, z$	$\mathrm{Im}\, z$	$\bar{z}$	$\lvert z\rvert$	$\mathrm{Arg}\, z$
$4i$	0	4	$-4i$	4	$\pi/2$
$1 + i\sqrt{3}$	1	$\sqrt{3}$	$1 - i\sqrt{3}$	2	$\pi/3$
$(1 - i)^3$	-2	-2	$(1 + i)^3 = -2 + 2i$	$2\sqrt{2}$	$-3\pi/4$

1.2 (i) A polar form is $4e^{i\pi/2}$. Square roots are $2e^{i\pi/4}$ and $-2e^{i\pi/4}$, that is $\pm\sqrt{2}(1+i)$.

(ii) A polar form is $2e^{i\pi/3}$. Square roots are $\sqrt{2}e^{i\pi/6}$ and $-\sqrt{2}e^{i\pi/6}$, that is $\pm\frac{1}{2}\sqrt{2}(\sqrt{3}+i)$.

(iii) A polar form is $2\sqrt{2}e^{-i3\pi/4}$. Square roots are $2^{3/4}e^{-i3\pi/8}$ and $-2^{3/4}e^{-i3\pi/8}$, that is $\pm 2^{3/4}\left[\cos\frac{3\pi}{8} - i\sin\frac{3\pi}{8}\right]$.

1.3



1.4 (i) Necessarily true—since if $z = x + iy$, $|y| \leq \sqrt{x^2 + y^2}$.

(ii) Necessarily true—since $(x - y)^2 \geq 0$, and so $2xy \leq x^2 + y^2$.

(iii) Necessarily true—since $|z + w|^2 = (z + w)(\bar{z} + \bar{w})$. (Note that using the result $|z|^2 = z\bar{z}$ is often quicker than working in terms of x and y.)

(iv) Sometimes true and sometimes false. This follows from part (iii) and the fact that $z\bar{w} + \bar{z}w$ can be positive or negative. (For example, let $z = w = 1$, and $z = i$, $w = -i$.)

(v) Sometimes true and sometimes false. (The correct inequality is $|z - w| \geq \big||z| - |w|\big|$; the given statement is therefore true only if the two sides are equal, for example, if $z = kw$ for some real positive number k.)

1.5 Since $\left(\frac{z+1}{z-1}\right)^7 = 1$, we have $\frac{z+1}{z-1} = e^{2k\pi i/7}$ $(k = 1, 2, \ldots, 6)$. Hence, rearranging, we obtain

$$z = \frac{e^{2k\pi i/7} + 1}{e^{2k\pi i/7} - 1} = \frac{\cos\frac{2k\pi}{7} + 1 + i\sin\frac{2k\pi}{7}}{\cos\frac{2k\pi}{7} - 1 + i\sin\frac{2k\pi}{7}} \qquad (k = 1, 2, \ldots, 6).$$

This can then be simplified further if desired.

(Note: We do not need to include $k = 0$, because $\frac{z+1}{z-1}$ has no solutions in C.

Unit 2 Continuous Functions

In this unit we introduced the concept of a function, and showed how functions can be represented pictorially. We also introduced the definitions of a limit and a continuous

function, and investigated their properties. The concepts of open and closed set were described in terms of cluster points, and the unit concluded with a discussion of connected sets, polygonally-connected sets and compact sets.

2.1 State the definitions of the following terms:

(i) the set T is the *inverse image* of the set S under the function f;

(ii) the function f has *limit* l *near* a;

(iii) the function f is *continuous at* a;

(iv) a is an *interior point* of the set S;

(v) S is an *open set*;

(vi) a is a *cluster point* of the set S;

(vii) S is a *closed set*;

(viii) the open set S is *connected*;

(ix) the set S is *polygonally-connected*;

(x) S is a *region*;

(xi) S is a *compact set.*

2.2 Which of the sets of points in Question 1.3 are (a) open, (b) closed, (c) a region, (d) polygonally-connected, (e) compact? (You are *not* asked to prove your answers.)

2.3 For each of the following functions f, find (a) its domain, (b) its range, (c) the image under f of the imaginary axis.

(i) $f:z \to z^4$ (ii) $f:z \to z^{-2}$ (iii) $f:z \to \dfrac{z-1}{z+1}$

2.4 For each of the functions f in Question 2.3, find the inverse image of the upper half-plane.

Solutions

2.1 See the following pages in *Unit 2*:

(i) 54, (ii) 59, (iii) 61, (iv) 62, (v) 63, (vi) 69, (vii) 70, (viii) 77, (ix) 80, (x) 81, (xi) 89.

2.2 (i) $\{z:\mathrm{Re}\, z > -1\}$ is open, a region, and polygonally-connected.

(ii) $\{z:|\mathrm{Im}\, z| > 1\}$ is open.

(iii) $\{z:1 < \mathrm{Re}\, z < 2 \text{ and } 1 \leq \mathrm{Im}\, z \leq 3\}$ is polygonally-connected.

(iv) $\{z:\mathrm{Re}\,(z^2) = 0\}$ is closed and polygonally-connected.

(v) $\left\{z:|z| \geq 1, |\mathrm{Arg}\, z| \leq \dfrac{\pi}{4}\right\}$ is closed and polygonally-connected.

2.3 (i) (a) $\mathbf{C}$ (b) $\mathbf{C}$

(c) The non-negative real axis (that is, $\{z:z = x + i0, x \geq 0\}$).

(ii) (a) $\mathbf{C} - \{0\}$ (b) $\mathbf{C} - \{0\}$

(c) The negative real axis (that is, $\{z:z = x + i0, x < 0\}$).

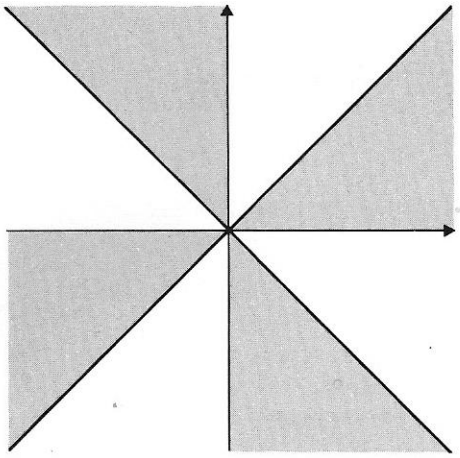
(iii) (a) $\mathbf{C} - \{-1\}$ (b) $\mathbf{C} - \{1\}$

(c) It is easy to verify that the point $z = ki$ is mapped to the point $x + iy$, where $x = \dfrac{k^2-1}{k^2+1}$ and $y = \dfrac{2k}{k^2+1}$. The set

$$\left\{z = x + iy : x = \frac{k^2-1}{k^2+1}, y = \frac{2k}{k^2+1}\right\}$$

is the unit circle $|z| = 1$ (apart from the point 1), as you might like to verify. So the image is the unit circle with the exception of the point 1.

2.4 (i) $\left\{z = re^{i\theta}: -\pi \le \theta \le -\frac{3\pi}{4} \text{ or } -\frac{\pi}{2} \le \theta \le -\frac{\pi}{4} \text{ or } 0 \le \theta \le \frac{\pi}{4} \text{ or } \frac{\pi}{2} \le \theta \le \frac{3\pi}{4}\right\}$



(ii) $\left\{z = re^{i\theta}: -\frac{\pi}{2} \le \theta \le 0 \text{ or } \frac{\pi}{2} \le \theta \le \pi\right\}$

(iii) If $z = x + iy$, then $\dfrac{z-1}{z+1} = \dfrac{(x-1)+iy}{(x+1)+iy} = \dfrac{(x^2+y^2-1)+i2y}{(x+1)^2+y^2}$.

If $\operatorname{Im}\left(\dfrac{z-1}{z+1}\right) \ge 0$, then $y \ge 0$.

It follows that the inverse image in question is the upper half-plane $\{z: \operatorname{Im} z \ge 0\}$.

Unit 3 Differentiation

In this unit we introduced the concept of differentiability and we interpreted the derivative as a scale factor. We then defined the functions exp, sin, cos, the hyperbolic functions, and the logarithm and power functions, paying particular attention to the domains of these functions and their derivatives. The unit concluded with a discussion of the Cauchy–Riemann equations, and their use in deciding whether or not a given function is differentiable.

3.1 Classify the following statements as true or false.

(i) The function $z \longrightarrow |z|$ is differentiable everywhere in $\mathbf{C}$.

(ii) If f is continuous at a, then f is differentiable at a.

(iii) If f is differentiable at a, and g is differentiable at $f(a)$, then the composite function $g \circ f$ is differentiable at a.

3.2 (i) What is the angle of intersection of the images of the real and imaginary axes under the function $f: z \longrightarrow (z+1)^2$?

(ii) Explain why the method you used in part (i) is not applicable to the function $f: z \longrightarrow z^2$. What are the images in this case?

3.3 Complete the following statements.

(i) $\dfrac{d}{dz}(\exp(e^z)) =$ _____

(ii) $(e^{iz} - e^{-iz})/\sin z =$ _____

(iii) $\cosh z = 0$ if and only if $z =$ _____

(iv) The domain of Log is _____

(v) $\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2 +$ _____

(vi) $\operatorname{Log} i =$ _____

3.4 (i) State the Cauchy–Riemann equations. Are they necessary for the existence of a derivative? Are they sufficient?

(ii) What can you say about the function $z \longrightarrow e^{-x}(\cos y - i \sin y)$ in the light of your answers to part (i)?

Solutions

3.1 (i) False.

(ii) False, in general; for example, $z \longrightarrow |z|$ is continuous at all points of $\mathbf{C}$, but differentiable at no points of $\mathbf{C}$.

(iii) True (Theorem 5 of *Unit 3*).

3.2 (i) We use the result: angles of intersection are preserved by an analytic function f at all points z_0 where $f'(z_0) \neq 0$. If $f(z) = (z+1)^2$, then $f'(0) = 2$, which is certainly non-zero, so that the images intersect at right-angles.

(ii) If $f(z) = z^2$, then $f'(0) = 0$, so we cannot use the above result. The image of the real axis is $\{w = f(z) : \operatorname{Im} w = 0, \operatorname{Re} w \geq 0\}$. The image of the imaginary axis is $\{w = f(z) : \operatorname{Im} w = 0, \operatorname{Re} w \leq 0\}$.

3.3 (i) $e^z \cdot \exp(e^z)$

(ii) $2i$

(iii) $\frac{1}{2}i\pi + ni\pi$ where n is an integer

(iv) $\{z : z$ is not zero or a negative real number$\}$

(v) $\pm 2\pi i$ or 0

(vi) $\frac{1}{2}\pi i$

3.4 (i) If $U = \operatorname{Re} f$, $V = \operatorname{Im} f$, where f is differentiable at $z_0 = x_0 + iy_0$, then, considering U and V as functions of two real variables x and y, we have

$$\frac{\partial U}{\partial x}(x_0, y_0) = \frac{\partial V}{\partial y}(x_0, y_0) \quad \text{and} \quad \frac{\partial V}{\partial x}(x_0, y_0) = -\frac{\partial U}{\partial y}(x_0, y_0).$$

These are the Cauchy–Riemann equations.

These conditions are necessary for the existence of the derivative of f at z_0; they are not, in general, sufficient, unless the partial derivatives $\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}$ are all continuous at z_0.

$$\left(\text{The derivative is given by } f'(z_0) = \frac{\partial U}{\partial x}(x_0, y_0) + i\frac{\partial V}{\partial x}(x_0, y_0).\right)$$

(ii) We have $U(x, y) = e^{-x}\cos y$, $V(x, y) = -e^{-x}\sin y$, so that

$$\frac{\partial U}{\partial x} = -e^{-x}\cos y, \qquad \frac{\partial U}{\partial y} = -e^{-x}\sin y,$$
$$\frac{\partial V}{\partial x} = e^{-x}\sin y, \qquad \frac{\partial V}{\partial y} = -e^{-x}\cos y.$$

So the Cauchy–Riemann equations are satisfied. Since all of these partial derivatives are continuous everywhere, the function

$$z \longrightarrow e^{-x}(\cos y - i \sin y)$$

is differentiable everywhere, that is, it is entire. (Its derivative is $-e^{-x}\cos y + ie^{-x}\sin y = -e^{-z}$.)

Unit 4 Integration

With this unit we began our study of integration. After an initial section on the properties of complex-valued functions of a real variable, we introduced the concept of an arc, and showed how to define the integral of a continuous function along a smooth

arc. Since this is independent of the parametrization chosen, we were able to define integrals along directed paths and contours. We proved the Fundamental Theorem for Contour Integrals, and we concluded the unit by proving the Estimation Theorem for the modulus of the integral of a function along a contour.

4.1 Which of the following arcs are

(a) simple, (b) closed, (c) simple-closed, (d) smooth?

(i) $\gamma_1(t) = e^{3it}$, $0 \leq t \leq 2\pi$.

(ii) $\gamma_2(t) = e^{3it}$, $0 \leq t \leq 3\pi$.

(iii) $\gamma_3(t) = t^2 + it^4$, $-1 \leq t \leq 1$.

Draw diagrams to show the paths of these arcs.

4.2 Find the integral of $z \longrightarrow z + 1$ along each of the arcs in Question 4.1.

4.3 Evaluate, in two different ways,

$$\int_\Gamma z\,dz,$$

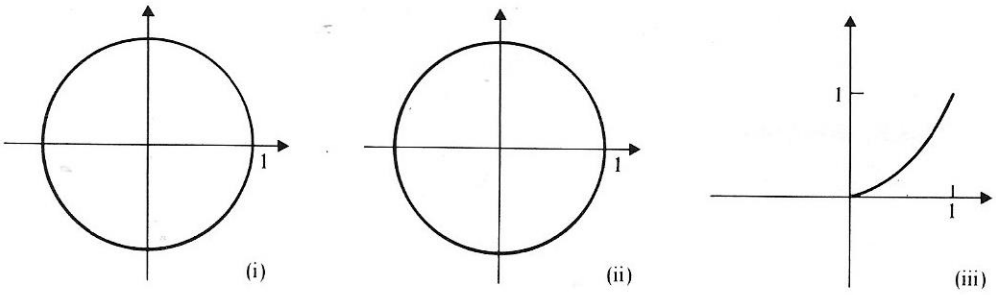
where Γ is the line-segment $[-i, 2]$.

4.4 Use the Estimation Theorem to show that, if C is the circle $|z| = 2$, then

$$\left| \int_C \frac{z^2 - 4z + 3}{(z^2 + 5)(z^2 - 2)} dz \right| \leq 30\pi.$$

Solutions

4.1 (i) Closed and smooth (ii) Smooth (iii) Closed



(i) (ii) (iii)

4.2 (i) $$\int_{\gamma_1} (z+1)dz = \int_0^{2\pi} (e^{3it} + 1) \cdot 3ie^{3it} dt = 3\left(\frac{e^{6it}}{6} + \frac{e^{3it}}{3}\right)\Big|_0^{2\pi} = 0$$

(ii) $$\int_{\gamma_2} (z+1)dz = \int_0^{3\pi} (e^{3it} + 1) \cdot 3ie^{3it} dt = 3\left(\frac{e^{6it}}{6} + \frac{e^{3it}}{3}\right)\Big|_0^{3\pi} = -2$$

(iii) Strictly speaking, we cannot integrate along γ_3 since it is not a smooth arc ($\gamma_3'(0) = 0$). However, this is a trivial difficulty and we overcome it by splitting γ_3 into the two arcs corresponding to $-1 \leq t \leq 0$ and $0 \leq t \leq 1$. Then we can integrate along each of these and add the results:

$$\begin{aligned}\int_{\gamma_3} (z+1)dz &= \int_{-1}^{0} (t^2 + it^4 + 1) \cdot (2t + 4it^3)dt \\ &\quad + \int_0^1 (t^2 + it^4 + 1) \cdot (2t + 4it^3)dt \\ &= \int_{-1}^{1} (t^2 + it^4 + 1) \cdot (2t + 4it^3)dt \\ &= 0, \quad \text{since each of the terms in this integral is an odd function.}\end{aligned}$$

4.3 *Method 1*

The line-segment $[-i, 2]$ can be parametrized by $\gamma(t) = -i + t(2 + i), 0 \leq t \leq 1$, so that

$$\int_\Gamma z\,dz = \int_0^1 [-i + t(2 + i)] \cdot (2 + i)dt$$

$$= (2 + i)(-it + \tfrac{1}{2}t^2(2 + i))\Big|_0^1$$

$$= (2 + i)(-i + \tfrac{1}{2}(2 + i)) = \tfrac{5}{2}.$$

Method 2

By the Fundamental Theorem for Contour Integrals (Theorem 8 of *Unit 4*), we have

$$\int_\Gamma z\,dz = \tfrac{1}{2}z^2\Big|_{-i}^2 = \tfrac{5}{2}.$$

4.4 On the circle $|z| = 2$,

$$|z^2 - 4z + 3| \leq 2^2 + 4 \cdot 2 + 3 = 15,$$
$$|z^2 + 5| \geq 5 - 2^2 = 1$$

and

$$|z^2 - 2| \geq 2^2 - 2 = 2.$$

It follows that

$$\left|\int_C \frac{z^2 - 4z + 3}{(z^2 + 5)(z^2 - 2)}dz\right| \leq \frac{15}{1 \cdot 2} \cdot (\text{length of } C) = 30\pi.$$

Unit 5 Cauchy's Theorem I

This unit was devoted to Cauchy's Theorem and some of its consequences. After introducing star regions, we proved the Cauchy–Goursat Theorem concerning the integral of an analytic function along a triangular contour. This led to Cauchy's Theorem for star regions, and Cauchy's Formulas (for a circle). As consequences of these results, we proved the theorems of Morera and Liouville and the Fundamental Theorem of Algebra, and concluded by discussing various techniques such as integration by parts and by substitution.

5.1 Which of the following sets are star sets:

(i) the interior of an ellipse;

(ii) the complex plane with the positive real axis removed;

(iii) the closed annulus bounded by the circles $|z| = 1$ and $|z| = 3$.

5.2 State:

(i) the Cauchy–Goursat Theorem;

(ii) Cauchy's Formula (for a circle);

(iii) Cauchy's Formula for derivatives (for a circle);

(iv) Morera's Theorem;

(v) Liouville's Theorem.

5.3 Use Cauchy's Formulas to evaluate:

(i) $\displaystyle\int_C \frac{\sin z}{z - 2}dz$, where C is the circle $|z| = 3$;

(ii) $\displaystyle\int_C \frac{\sin z}{(z - 2)^3}dz$, where C is the circle $|z| = 3$;

(iii) $\displaystyle\int_C \frac{\sin z}{z^2 - 2}dz$, where C is the circle $|z - 2| = 3$;

(iv) $\displaystyle\int_C \frac{\sin z}{z^2-2}\,dz$, where C is the circle $|z| = 3$.

(Note: In parts (iii) and (iv) you might feel that it is more appropriate to use the Residue Theorem. If so, and you have the time, do them by both methods.)

5.4 Evaluate

(i) $\displaystyle\int_0^{\pi i} z\cos z\,dz$, (ii) $\displaystyle\int_0^{\pi i} z\cos(z^2)\,dz$.

Solutions

5.1 (i) Yes, since it is convex.

(ii) Yes, the stars are the points on the non-positive real axis.

(iii) No, for obvious reasons.

5.2 See the following pages in *Unit 5*:
(i) 63, (ii) 74, (iii) 82, (iv) 88, (v) 89.

5.3 (i) Since $\sin 2 = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{\sin z}{z-2}\,dz$, the answer is $2\pi i \sin 2$.

(ii) Since $\sin''(2) = \dfrac{2!}{2\pi i}\displaystyle\int_C \frac{\sin z}{(z-2)^3}\,dz$, the answer is $-\pi i \sin 2$.

(iii) The point $\sqrt{2}$ lies inside the circle, so letting $f(z) = \dfrac{\sin z}{z+\sqrt{2}}$, we have

$$\frac{1}{2\pi i}\int_C \frac{\sin z}{z^2-2}\,dz = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-\sqrt{2}}\,dz = f(\sqrt{2}) = \frac{\sin\sqrt{2}}{2\sqrt{2}},$$

so that the answer is $(\pi i \sin\sqrt{2})/\sqrt{2}$.

(iv) Both of the points $\sqrt{2}$ and $-\sqrt{2}$ lie inside the circle C. We use partial fractions and write

$$\frac{1}{z^2-2} = \frac{1}{2\sqrt{2}}\left(\frac{1}{z-\sqrt{2}} - \frac{1}{z+\sqrt{2}}\right).$$

But $\sin\sqrt{2} = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{\sin z}{z-\sqrt{2}}\,dz$, and $\sin(-\sqrt{2}) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{\sin z}{z+\sqrt{2}}\,dz$, so that the answer is $\dfrac{2\pi i}{2\sqrt{2}}(\sin\sqrt{2} - \sin(-\sqrt{2})) = \pi i\sqrt{2}\sin\sqrt{2}$.

5.4 (i) Integrating by parts, we obtain

$$\begin{aligned}\int_0^{\pi i} z\cos z\,dz &= z\sin z\Big|_0^{\pi i} - \int_0^{\pi i} 1\cdot\sin z\,dz\\ &= (\pi i\sin\pi i - 0) + \cos z\Big|_0^{\pi i}\\ &= \pi i\sin\pi i + \cos\pi i - 1\\ &= -\pi\sinh\pi + \cosh\pi - 1.\end{aligned}$$

(ii) Substituting $w = z^2$, we obtain

$$\int_0^{\pi i} z\cos(z^2)dz = \int_0^{-\pi^2} \tfrac{1}{2}\cos w\,dw = \tfrac{1}{2}\sin w\Big|_0^{-\pi^2} = -\tfrac{1}{2}\sin(\pi^2).$$

Unit 6 Taylor Series

This unit was devoted to a discussion of the way we can use power series to represent analytic functions. We first stated analogues for complex numbers of various well-known results on sequences and series of real numbers, and then introduced the concept of a complex power series. We defined the radius and circle of convergence of such a power series, and showed how power series can be used to define analytic functions. We also proved Taylor's Theorem, which tells us that an analytic function can be represented by power series. Finally, we proved the Uniqueness Theorem, and used it to derive results on particular complex series from the analogous results for real series.

6.1 Show that the functions given by

$$f(z) = \sum_{n=0}^{\infty} \frac{(2z)^{2n+1}}{(2n+1)!} \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} \frac{(2z)^{2n}}{(2n)!}$$

are defined for all $z \in \mathbf{C}$, and that $f' = 2g$ and $g' = 2f$.

6.2 Prove that $|e^z - 1| \leq e^{|z|} - 1$ for all $z \in \mathbf{C}$.

6.3 For each of the following functions f and points α, find the first three non-zero terms in the Taylor series for f at α:

(i) $f(z) = \dfrac{e^{2z}}{1+z}$, $\alpha = 0$;

(ii) $f(z) = \dfrac{1}{z^2}$, $\alpha = 2$;

(iii) $f(z) = \dfrac{z+1}{\cos z}$, $\alpha = 0$.

6.4 (a) Find the radius of convergence of each of the Taylor series in Question 6.3.

(b) If $\sum_{n=0}^{\infty} a_n$ is convergent but $\sum_{n=0}^{\infty} (-1)^n a_n$ is divergent, what is the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$?

6.5 (i) State the Uniqueness Theorem.

(ii) Use the Uniqueness Theorem to prove that $\sin 2z = 2 \sin z \cos z$ for all $z \in \mathbf{C}$.

Solutions

6.1 Applying the ratio test (*Unit 6*, page 110) to f, we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{4z^2}{2n(2n+1)} \right| = 0,$$

so that the series is defined for all $z \in \mathbf{C}$. A similar argument holds for g. If we now differentiate term by term (which is permissible by Theorem 8 on page 120 of *Unit 6*), we obtain

$$f'(z) = 2g(z) \quad \text{and} \quad g'(z) = 2f(z).$$

(The functions f and g are, of course, $z \longrightarrow \sinh 2z$ and $z \longrightarrow \cosh 2z$, respectively.)

6.2 $$|e^z - 1| = \left| \sum_{n=1}^{\infty} \frac{z^n}{n!} \right| \leq \sum_{n=1}^{\infty} \frac{|z|^n}{n!} = e^{|z|} - 1$$

6.3 (i) $$\frac{e^{2z}}{1+z} = e^{2z} \cdot (1+z)^{-1}$$

$$= \left(1 + 2z + \frac{(2z)^2}{2!} + \ldots\right) \cdot (1 - z + z^2 - \ldots)$$

$$= 1 + z(2-1) + z^2(2-2+1) + \ldots$$

$$= 1 + z + z^2 + \ldots$$

(ii) $$\frac{1}{z^2} = \frac{1}{[(z-2)+2]^2} = \frac{1}{4}\left(1 + \frac{1}{2}(z-2)\right)^{-2}$$
$$= \frac{1}{4}\left[1 - 2\cdot\frac{1}{2}(z-2) + 3\left(\frac{1}{2}(z-2)\right)^2 - \ldots\right]$$
$$= \frac{1}{4} - \frac{1}{4}(z-2) + \frac{3}{16}(z-2)^2 - \ldots$$

(iii) $$\frac{z+1}{\cos z} = (1+z)\cdot\left(1 - \frac{z^2}{2} + \ldots\right)^{-1}$$
$$= (1+z)\cdot\left(1 + \frac{z^2}{2} + \ldots\right)$$
$$= 1 + z + \frac{1}{2}z^2 + \ldots$$

(It is also possible to use the method of comparing coefficients: see Section 6.6 of *Unit 6*.)

6.4 (a) (i) 1, because the function is not analytic on any region containing $z = -1$.

(ii) 2, because the function is not analytic on any region containing $z = 0$.

(iii) $\frac{\pi}{2}$, because the function is not analytic on any region containing $z = \pm\frac{\pi}{2}$.

(b) Let the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ be r. Since the series converges if $z = 1$, it follows that $r \geq 1$. Since the series diverges if $z = -1$, it follows that $r \leq 1$. Hence $r = 1$.

6.5 (i) Let f and g be analytic on a region R, let S be a subset of R with a cluster point in R, and let $f = g$ on S. Then $f = g$ on R.

(ii) Take R to be the complex plane, S to be the real line, f to be the function $z \longrightarrow \sin 2z$, and g to be the function $z \longrightarrow 2\sin z \cos z$. Then $f = g$ on S (since $\sin 2x = 2\sin x \cos x$ for all *real* x), and the result follows.

Unit 8 Singularities

In this unit we look at the types of singularity in $\mathbf{C}$ which a function can have. We began by defining removable singularities, poles and essential singularities, and investigated the behaviour of a function near each of these types of singularity; in particular, we proved the Casorati–Weierstrass Theorem, which describes the behaviour of a function near an essential singularity. We then showed how the type of singularity determines the form of the Laurent series, and we ended by using residue methods to evaluate some simple integrals.

8.1 Define the following terms.

(i) f has a *singularity at* α.

(ii) f has a *pole of order* m *at* α.

(iii) f has a *removable singularity at* α.

(iv) f has an *essential singularity at* α.

What form does the Laurent series for f about the point α take in each case?

8.2 Find all of the singularities of the following functions, and classify them as removable singularities, poles or essential singularities. Give the order of any poles.

(i) $z \longrightarrow \dfrac{1}{(z+1)(z-3)^2}$ (ii) $z \longrightarrow \dfrac{z}{(z^2+1)^3}$

(iii) $z \longrightarrow \dfrac{e^z}{z^4}$ (iv) $z \longrightarrow \dfrac{\sin z}{z^4}$

(v) $z \longrightarrow \dfrac{z^4}{\sin z}$ (vi) $z \longrightarrow \dfrac{\sin^2 z}{z^3}$

(vii) $z \longrightarrow \dfrac{z^3}{\sin^2 z}$ (viii) $z \longrightarrow \dfrac{1}{z} e^{-z}$

(ix) $z \longrightarrow ze^{1/z}$ (x) $z \longrightarrow \cot z$

8.3 For each of the following functions f and points α, find the first three non-zero terms in the Laurent series for f about α.

(i) $z \longrightarrow \dfrac{e^z}{z^4}$, $\alpha = 0$.

(ii) $z \longrightarrow \dfrac{1}{(z+1)(z-3)^2}$, $\alpha = 3$.

(iii) $z \longrightarrow \dfrac{z^4}{\sin z}$, $\alpha = 0$.

8.4 (i) Define the term *residue of f at α*.

(ii) Find the Laurent series for

$$f : z \longrightarrow \frac{1}{(z-1)(z-3)}$$

which is convergent on the annulus $1 < |z| < 3$.

What is the residue of f at 0?

Solutions

8.1 (i) The function f has a singularity at α if f is analytic on some punctured disc $0 < |z - \alpha| < r$, but not on the open disc $|z - \alpha| < r$. The Laurent series in this case has the form

$$\ldots + \frac{a_{-2}}{(z-\alpha)^2} + \frac{a_{-1}}{z-\alpha} + a_0 + a_1(z-\alpha) + \ldots,$$

where at least one of the coefficients $a_0, a_{-1}, a_{-2}, \ldots$ is non-zero.

(ii) The function f has a pole of order m at α if f has a singularity at α and there is a function g, analytic on some neighbourhood of α, with $g(\alpha) \neq 0$, such that for $z \neq \alpha$,

$$f(z) = \frac{g(z)}{(z-\alpha)^m}, \quad \text{where } m \text{ is a positive integer.}$$

The Laurent series in this case has the form

$$\frac{a_{-m}}{(z-\alpha)^m} + \frac{a_{-m+1}}{(z-\alpha)^{m-1}} + \cdots + \frac{a_{-1}}{z-\alpha} + a_0 + a_1(z-\alpha) + \ldots$$

$$= \sum_{n=-m}^{\infty} a_n (z-\alpha)^n, \quad \text{where } a_{-m} \neq 0.$$

(iii) The function f has a removable singularity at α if f has a singularity at α, and there is some complex number a such that the function

$$z \longrightarrow \begin{cases} f(z), & \text{if } z \neq \alpha, \\ a, & \text{if } z = \alpha, \end{cases}$$

is analytic on a neighbourhood of α.

The Laurent series in this case has the form

$$a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots = \sum_{n=0}^{\infty} a_n(z-\alpha)^n.$$

(This is a Taylor series.)

(iv) The function f has an essential singularity at α if f has a singularity which is neither a removable singularity nor a pole.
The Laurent series in this case has the form

$$\ldots + \frac{a_{-2}}{(z-\alpha)^2} + \frac{a_{-1}}{z-\alpha} + a_0 + a_1(z-\alpha) + \cdots = \sum_{n=-\infty}^{\infty} a_n(z-\alpha)^n.$$

8.2 (i) Simple pole at -1, pole of order two at 3.

(ii) Poles of order three at i and $-i$.

(iii) Pole of order four at 0.

(iv) Pole of order three at 0.

(v) Simple poles at $\pm n\pi$ $(n = 1, 2, 3, \ldots)$, removable singularity at 0.

(vi) Simple pole at 0.

(vii) Poles of order two at $\pm n\pi$ $(n = 1, 2, 3, \ldots)$, removable singularity at 0.

(viii) Simple pole at 0.

(ix) Essential singularity at 0.

(x) Simple poles at $\pm n\pi$ $(n = 0, 1, 2, \ldots)$.

8.3 (i) $$\frac{e^z}{z^4} = \frac{1}{z^4}\left(1 + z + \frac{z^2}{2!} + \ldots\right) = \frac{1}{z^4} + \frac{1}{z^3} + \frac{1}{2z^2} + \ldots$$

(ii) $$\frac{1}{(z+1)(z-3)^2} = \frac{1}{(z-3)^2}[4 + (z-3)]^{-1}$$

$$= \frac{1}{4(z-3)^2}\left(1 - \frac{(z-3)}{4} + \frac{(z-3)^2}{4^2} - \ldots\right)$$

$$= \frac{1}{4(z-3)^2} - \frac{1}{16(z-3)} + \frac{1}{64} - \ldots$$

(iii) $$\frac{z^4}{\sin z} = z^3\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots\right)^{-1}$$

$$= z^3\left[1 - \left(\frac{z^2}{6} - \frac{z^4}{120} + \ldots\right)\right]^{-1}$$

$$= z^3\left[1 + \left(\frac{z^2}{6} - \frac{z^4}{120} + \ldots\right) + \left(\frac{z^2}{6} - \frac{z^4}{120} + \ldots\right)^2 + \ldots\right]$$

$$= z^3\left[1 + \frac{z^2}{6} + z^4\left(-\frac{1}{120} + \frac{1}{36}\right) + \ldots\right]$$

$$= z^3 + \frac{z^5}{6} + \frac{7z^7}{360} + \ldots$$

(It is also possible to use the method of comparing coefficients: see Section 8.8. of *Unit 8*.)

8.4 (i) If f has a singularity at α, the coefficient of $\dfrac{1}{(z-\alpha)}$ in its Laurent expansion about α is called the residue of f at α.

(ii) $$\frac{1}{(z-1)(z-3)} = \frac{1}{2}\left(\frac{1}{z-3} - \frac{1}{z-1}\right)$$

$z \longrightarrow \dfrac{1}{z-3}$ is analytic on the disc $|z| < 3$; its Laurent expansion there is

$$-\left(\frac{1}{3} + \frac{z}{3^2} + \frac{z^2}{3^3} + \frac{z^3}{3^4} + \ldots\right).$$

$z \longrightarrow \dfrac{1}{z-1}$ is analytic on the set $\{z:|z|>1\}$; its Laurent expansion there is

$$\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots.$$

The function f therefore has Laurent expansion

$$\ldots-\frac{1}{2z^3}-\frac{1}{2z^2}-\frac{1}{2z}-\frac{1}{6}-\frac{z}{2\cdot 3^2}-\frac{z^2}{2\cdot 3^3}-\frac{z^3}{2\cdot 3^4}-\ldots,$$

convergent on the annulus $1<|z|<3$.

The function f is analytic at 0 and so $\operatorname{Res}(f,0)=0$.

Unit 9 Cauchy's Theorem II

In this unit we returned to Cauchy's Theorem and its consequences. We started by proving the Maximum Principle and the Boundary Uniqueness Theorem. We then defined the term simply-connected region, and proved Cauchy's Theorem for such a region. This was followed by a discussion on various applications, and the unit concluded with the statement and proof of the Residue Theorem.

9.1 State:

(i) the Local Maximum Principle;

(ii) the Maximum Principle;

(iii) the Minimum Principle;

(iv) the Residue Theorem.

9.2 Find the maximum value of $|\exp(z^3)|$ on the disc $|z|\leq 2$.

9.3 (i) Define the terms (a) *simply-connected* and (b) *analytic logarithm of f on R.*

(ii) Determine which of the following regions are simply-connected and which are not.

(a) $S_1=\{z:|z|<1, z\neq 0\}$

(b) $S_2=\{z:|z|<1, \operatorname{Im} z<0\}$

(c) $S_3=\{z:1<|z|<2\}$.

9.4 (i) Define the term *winding number.*

(ii) Let Γ be a simple-closed contour and let α be a point not on Γ. What can you say about the value of $\operatorname{Wnd}(\Gamma,\alpha)$?

9.5 What are the possible values of $\displaystyle\int_\Gamma \frac{\cos z}{z^5}\,dz$, where Γ is a simple-closed contour, with positive orientation, not passing through 0?

Solutions

9.1 See the following pages in *Unit 9*:
(i) 60, (ii) 62, (iii) 65, (iv) 103.

9.2 By the Maximum Principle, the maximum value occurs on the boundary of the disc, that is at some point $z=2e^{i\theta}$. But

$$\exp(z^3)=\exp(8e^{3i\theta})=\exp(8\cos 3\theta+8i\sin 3\theta),$$

and so

$$|\exp(z^3)|=|e^{8\cos 3\theta}|\cdot|e^{8i\sin 3\theta}|=e^{8\cos 3\theta}.$$

Since the maximum value of $e^{\cos 3\theta}$ is e (which occurs when $\cos 3\theta=1$), the maximum value of $|\exp(z^3)|$ is e^8.

9.3 (i) See the following pages in *Unit 9*:
(a) 71 (b) 76.

(ii) (a) S_1 is not simply-connected: the square with vertices $\pm\frac{1}{2} \pm \frac{1}{2}i$ is a simple polygon in R whose inside is not contained in R.

(b) S_2 is simply-connected.

(c) S_3 is *not* simply-connected: the simple polygon with vertices at $\frac{3}{2}\exp(n\pi i/4)$ $n = 0, 1, \ldots, 7$ is in R, but its inside is not contained in R.

9.4 (i) Let Γ be a closed contour, and α a point not on Γ. The winding number of Γ about α is $\dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{dz}{z-\alpha}$.

(ii) By Theorem 8 of *Unit 9*,

$$\mathrm{Wnd}(\Gamma, \alpha) = \begin{cases} \pm 1, & \text{if } \alpha \text{ lies inside } \Gamma, \\ 0, & \text{if } \alpha \text{ lies outside } \Gamma. \end{cases}$$

9.5 Let $f(z) = \dfrac{\cos z}{z^5}$. f has a pole at 0, and since

$$\frac{\cos z}{z^5} = \frac{1}{z^5} - \frac{1}{2z^3} + \frac{1}{24z} - \ldots,$$

the residue of f at 0 is $\dfrac{1}{24}$. Hence

$$\int_\Gamma f = \begin{cases} \frac{\pi i}{12}, & \text{if 0 lies inside } \Gamma, \\ 0, & \text{if 0 lies outside } \Gamma. \end{cases}$$

Unit 10 The Calculus of Residues

This unit was devoted to some applications of the Residue Theorem. We started with a section in which we showed various different ways of calculating residues, and we demonstrated the use of the Residue Theorem in evaluating certain contour integrals, in particular real trigonometrical integrals. This was followed by a section on the evaluation of real improper integrals of the form $\displaystyle\int_{-\infty}^{\infty} f(t)\cos \beta t\, dt$ and $\displaystyle\int_{-\infty}^{\infty} f(t)\sin \beta t\, dt$, where $\beta \geq 0$, and f is a rational function. We also showed how these methods could be adapted to sum various infinite series. The unit concluded with a discussion of the Principle of the Argument.

10.1 Calculate the residues of the following functions f at the points α indicated.

(i) $z \longrightarrow \dfrac{z}{(z-1)(z-2)^2}$, $\alpha = 1$

(ii) $z \longrightarrow \dfrac{z}{(z-1)(z-2)^2}$, $\alpha = 2$

(iii) $z \longrightarrow e^z \operatorname{cosec} z$, $\alpha = 0$

(iv) $z \longrightarrow e^{1/z}$, $\alpha = 0$

10.2 Convert the following trigonometrical integral into a contour integral taken along the unit circle C:

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta + \sin\theta}.$$

(Do not bother to evaluate it.)

10.3 Describe briefly how you would use complex analysis to evaluate

(i) the real improper integral $\displaystyle\int_{-\infty}^{\infty} \frac{\cos t}{(t-2)(t^2+1)}\,dt$;

(ii) the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$.

10.4 Verify the Principle of the Argument in the form

$$N - P = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz,$$

where C is the circle $|z| = 2$ and $f(z) = \dfrac{(z-1)^2}{(z+1)(z+3)^2}$.

Solutions

10.1 (i) 1, by the cover-up rule.

(ii) -1, by writing $z - 2 = h$, and finding the coefficient of h^{-1}.

(iii) 1, by the "g/h rule" (Theorem 2 of *Unit 10*), with $g(z) = e^z$, $h(z) = \sin z$.

(iv) 1, since $e^{1/z} = 1 + \dfrac{1}{z} + \dfrac{1}{2!z^2} + \ldots$.

10.2 If $z = e^{i\theta}$, then $\dfrac{dz}{d\theta} = iz$, $\cos\theta = \dfrac{1}{2}(z + z^{-1})$ and $\sin\theta = \dfrac{1}{2i}(z - z^{-1})$, so that

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta + \sin\theta} = \int_C \frac{1}{2 + \frac{1}{2}(z + z^{-1}) + \frac{1}{2i}(z - z^{-1})} \cdot \frac{dz}{iz}$$

$$= \int_C \frac{2\,dz}{z^2(1+i) + 4iz + (i-1)}$$

10.3 (i) Consider the integral $\displaystyle\int_\Gamma \frac{e^{iz}}{(z-2)(z^2+1)}\,dz$, where the contour Γ consists of the real axis from $-r$ to r (with a small semicircular indentation to avoid the pole at 2), and a semicircle of radius r, centre the origin, in the upper half-plane. The Estimation Theorem tells us that the integral along the large semicircle has limit zero as r becomes large. The value of the contour integral is therefore

$$2\pi i \times (\text{residue at } i) + \pi i \times (\text{residue at } 2).$$

The result follows on taking real parts.

(ii) Let $g(z) = \dfrac{\pi \cot \pi z}{z^2+4}$, and consider the integral $\displaystyle\int_{S_N} g(z)dz$, where S_N is the square contour with vertices $(N + \frac{1}{2})(1 + i)$, $(N + \frac{1}{2})(-1 + i)$, $(N + \frac{1}{2})(-1 - i)$ and $(N + \frac{1}{2})(1 - i)$. Then, by the Residue Theorem,

$$\int_{S_N} g(z)dz = 2\pi i \left(\text{Res}\,(g, 2i) + \text{Res}\,(g, -2i) + \text{Res}\,(g, 0) + 2\sum_{n=1}^{N} \frac{1}{n^2+4}\right),$$

since $z \longrightarrow 1/(z^2+4)$ is an even function. If we now let N become large, then the value of the integral can easily be shown to have limit zero. It follows that

$$\sum_{n=1}^{\infty} \frac{1}{n^2+4} = -\frac{1}{2}(\text{Res}\,(g, 2i) + \text{Res}\,(g, -2i) + \text{Res}\,(g, 0)).$$

10.4 If $f(z) = \dfrac{(z-1)^2}{(z+1)(z+3)^2}$, then $\dfrac{f'(z)}{f(z)} = \dfrac{-z^2+6z+11}{(z-1)(z+1)(z+3)}$.

By the Residue Theorem,

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \text{Res}\,(f'/f, 1) + \text{Res}\,(f'/f, -1)$$

$$= \frac{16}{2\cdot 4} + \frac{4}{(-2)\cdot 2} = 1.$$

But $N - P = 2 - 1 = 1$, as required.

Unit 11 Analytic Functions

We started the unit with a discussion of uniform convergence, and proved such results as the Weierstrass M-test, and the theorem which allows us to "differentiate under the integral sign". This was followed by a discussion of analytic continuation, and the definition of an analytic multi-function. This led on to a brief discussion of Riemann surfaces and complete analytic functions: we showed that the infinite spiral surface could be taken as the Riemann surface of both the (complete analytic) logarithm function and the (complete analytic) power functions $z \longrightarrow z^\alpha$. We concluded with a section on applications, including Rouché's Theorem, contour integrals, the gamma function, the Schwarz Reflection Principle, and the method of analytic continuation by means of Taylor series.

11.1 State the Weierstrass M-test, and use it to show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + |z|^3}$$

converges uniformly on $\mathbf{C}$.

11.2 (i) What does it mean to say that two function elements are direct analytic continuations of each other?

(ii) What does it mean to say that the infinite spiral surface is a Riemann surface for the logarithm function?

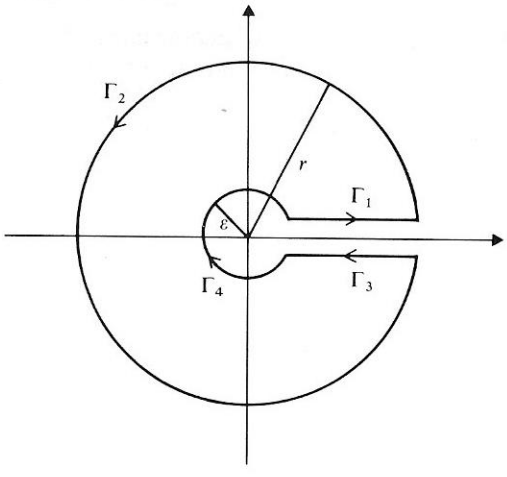
(iii) What is the difference between an analytic multi-function and a complete analytic function?

11.3 (i) State Rouché's Theorem.

(ii) Let $f_N(z) = \sum_{n=0}^{N} \dfrac{z^n}{n!}$. Show that for sufficiently large N, f_N has no zeros in the disc $|z| < 1$.

11.4 Let a be a real number such that $0 < a < 1$. By considering the integral of the complete analytic function $z \longrightarrow \dfrac{z^{a-1}}{1+z}$ along the contour shown, evaluate

$$I = \int_0^\infty \frac{x^{a-1}}{1+x}dx.$$



11.5 Let $f(z) = z - \frac{z^2}{2^2} + \frac{z^3}{3^2} - \cdots + (-1)^{n+1}\frac{z^n}{n^2} + \ldots$, where $|z| < 1$. Show that the series converges for *all* z in the closed disc $|z| \leq 1$, and show that the only barrier point of the series is at $z = -1$.

Solutions

11.1 Let $\{f_n\}$ be a sequence of functions defined on a set A, and suppose that $\{M_n\}$ is a sequence of real numbers such that $|f_n(z)| \leq M_n$ for all z in A. Suppose also that $\sum_{n=1}^{\infty} M_n$ converges. Then for each z in A, the series $\sum_{n=1}^{\infty} f_n(z)$ converges and $\sum_{n=1}^{\infty} f_n$ converges uniformly on A to the function $z \longrightarrow \sum_{n=1}^{\infty} f_n(z)$. To prove the last part, we use the Weierstrass M-test with $f_n(z) = 1/(n^3 + |z|^3)$ and $M_n = 1/n^3$. The result follows since $1/(n^3 + |z|^3) \leq 1/n^3$, and $\sum_{n=1}^{\infty} 1/n^3$ converges.

11.2 (i) Two function elements (f_1, R_1) and (f_2, R_2) are direct analytic continuations of each other if $f_1(z) = f_2(z)$ for all z in some non-empty open subset of $R_1 \cap R_2$.

(ii) It means that we can define the (complete analytic) logarithm function in such a way that its domain is the spiral surface. For any point z_0 on the surface, we choose a neighbourhood R, and then find the corresponding function element (L, R) of the logarithm; the value of the logarithm function at z_0 is then defined to be $L(z_0)$.

(iii) An analytic multi-function is an equivalence class of function elements, where the equivalence relation $\sim$ is:

$(f, R) \sim (g, S)$ if and only if (f, R) is an analytic continuation of (g, S).

A complete analytic function is the interpretation of an analytic multi-function as a function with a suitable Riemann surface as its domain.

11.3 (i) Let f and g be analytic on a simply-connected region R, and let Γ be a simple-closed contour in R. If $|f(z)| > |g(z)|$ for each z on Γ, then f and $f + g$ have the same number of zeros inside Γ (each counted according to its order).

(ii) Let $g_N(z) = \sum_{n=N+1}^{\infty} \frac{z^n}{n!}$ and C be the circle $|z| = 1$. We shall show that $|f_N(z)| > |g_N(z)|$ for $z \in C$ and N sufficiently large. Since $\lim_{N\to\infty} \sum_{n=0}^{N} \frac{z^n}{n!} = \exp z$, given $\varepsilon > 0$ there exists N_0 such that if $N > N_0$ then $|f_N(z) - \exp z| < \varepsilon$ for all z. But $\exp z = f_N(z) + g_N(z)$, and so for all z

$$|g_N(z)| < \varepsilon \text{ if } N > N_0.$$

Let $z = x + iy$; then $|\exp z| = e^x$. So if z is on C (that is, $|z| = 1$) we have $|\exp z| \geq e^{-1}$, and so

$$|f_N(z) + g_N(z)| \geq e^{-1}.$$

It follows that

$$|f_N(z)| + |g_N(z)| \geq e^{-1}.$$

But if $N > N_0$, we have $|g_N(z)| < \varepsilon$ and so if $N > N_0$ and $|z| = 1$, we have

$$|f_N(z)| \geq e^{-1} - \varepsilon.$$

Let N_1 be such that $|g_N(z)| < \varepsilon < e^{-1}/2$ if $N > N_1$. Then if $|z| = 1$ and $N > N_1$,

$$|f_N(z)| > |g_N(z)|.$$

We now apply Rouché's Theorem. Since $\exp = f_N + g_N$ has no zeros (anywhere), and $|f_N(z)| > |g_N(z)|$ on C, f_N has no zeros in the disc $|z| < 1$ provided N is sufficiently large.

11.4 Put $f(z) = z^{a-1}/(1+z)$, which has a branch point at 0. (Remember that $z^{a-1} = \exp\{(a-1)\log z\}$.)

We use the contour $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$, which we suppose is shown on the Riemann surface S for log. (This contour is carefully chosen to avoid the branch point at 0, and both Γ_1 and Γ_3 correspond to the interval $[\varepsilon, r]$ of the real axis; however, the points of Γ_1 have argument 0, and the points of Γ_3 have argument 2π.

The function $f(z)$ has a pole at the point $e^{i\pi}$ of S with residue $(e^{i\pi})^{a-1} = e^{ia\pi}e^{-i\pi} = -e^{ia\pi}$; and this pole is inside Γ if $0 < \varepsilon < 1 < r$. Hence, applying the Residue Theorem, we obtain

$$\begin{aligned}\int_\Gamma \frac{z^{a-1}}{1+z}dz &= \int_{\Gamma_1} f + \int_{\Gamma_2} f + \int_{\Gamma_3} f + \int_{\Gamma_4} f \\ &= \int_\varepsilon^r \frac{x^{a-1}}{1+x}dx + \int_0^{2\pi} \frac{(re^{i\theta})^{a-1} ire^{i\theta}}{1+re^{i\theta}}d\theta \\ &\quad + \int_r^\varepsilon \frac{(xe^{2\pi i})^{a-1}}{1+x}dx + \int_{2\pi}^0 \frac{(\varepsilon e^{i\theta})^{a-1} i\varepsilon e^{i\theta}}{1+\varepsilon e^{i\theta}}d\theta \\ &= 2\pi i \operatorname{Res}(f, e^{i\pi}) \\ &= -2\pi i e^{ia\pi}.\end{aligned}$$

The first and third integrals combine to give

$$\begin{aligned}\{1-(e^{2\pi i})^{a-1}\}\int_\varepsilon^r \frac{x^{a-1}}{1+x}dx &= \{1-e^{2\pi ai}\}\int_\varepsilon^r \frac{x^{a-1}}{1+x}dx \\ &= -e^{\pi ai}\{e^{\pi ai} - e^{-\pi ai}\}\int_\varepsilon^r \frac{x^{a-1}}{1+x}dx \\ &= -2ie^{\pi ai}\sin(\pi a)\int_\varepsilon^r \frac{x^{a-1}}{1+x}dx.\end{aligned}$$

Now, for the second integral,

$$\left|\int_{\Gamma_2} f\right| \le \frac{2\pi r^a}{r-1} \qquad (\text{since } r > 1),$$

so that $\int_{\Gamma_2} f \longrightarrow 0$ as $r \longrightarrow \infty$, since $a < 1$.

And for the fourth integral,

$$\left|\int_{\Gamma_4} f\right| \le \frac{2\pi\varepsilon^a}{1-\varepsilon} \qquad (\text{since } \varepsilon < 1)$$

so that $\int_{\Gamma_4} f \longrightarrow 0$ as $\varepsilon \longrightarrow 0$, since $a > 0$.

Hence if we let $r \longrightarrow \infty$ and $\varepsilon \longrightarrow 0$ in our previous identity, we obtain

$$-2ie^{\pi ai}\sin(\pi a)I = -2\pi i e^{ia\pi};$$

that is,

$$I = \frac{\pi}{\sin(\pi a)}.$$

11.5 Notice that $\left|(-1)^{n+1}\frac{z^n}{n^2}\right| \le \frac{1}{n^2}$ if $|z| \le 1$, and since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, the given series converges uniformly for all z in the disc $|z| \le 1$, by the Weierstrass M-test. Since a Taylor series may be differentiated term by term inside its circle of convergence, we have, for $|z| < 1$,

$$\begin{aligned}f'(z) &= 1 - \frac{z}{2} + \frac{z^2}{3} - \cdots + (-1)^{n+1}\frac{z^{n-1}}{n} + \ldots \qquad (*) \\ &= \frac{1}{z}\operatorname{Log}(1+z).\end{aligned}$$

But $z \longrightarrow (1/z)\operatorname{Log}(1+z)$ has a removable singularity at 0 and a branch point at -1. Thus -1 is a barrier point for the series for $f'(z)$. There are no other barrier points, because $z \longrightarrow (1/z)\operatorname{Log}(1+z)$ provides a direct analytic continuation of f' (the function element defined by the series $(*)$). The series for $f'(z)$ has -1 as its only barrier point and this implies that the same is true for the original series for $f(z)$.

R.2 OPEN-ENDED QUESTIONS

In this section we shall pose a number of open-ended questions which we hope will enable you to test your overall understanding of the course. You may wish to make notes before referring to our solutions, but this is not really necessary. For nearly all of the questions it will probably be sufficient for you to think for two or three minutes about your answer and then to look at ours. But do try to formulate your own answers; these questions will do you little good if you simply go straight to the solutions. The answers are grouped together after the questions.

1. Requiring a complex function to be differentiable at a point is a stronger condition than requiring a real function to be differentiable. Illustrate this fact and state some consequences of it.

2. The concept of compactness has played a significant role in this course. Give some examples of its use.

3. Give an example of a real function which is one–one but whose complex analogue is not one–one. The existence of such functions creates a problem—what is it, and how did we deal with it?

4. List some of the similarities and differences between the functions $\sin: \mathbf{R} \longrightarrow \mathbf{R}$ and $\sin: \mathbf{C} \longrightarrow \mathbf{C}$.

5. Outline the main steps in the formulation of the definition of contour integral.

6. "Cauchy's Theorem is the central theorem of the course." Illustrate this point of view by listing some results which depend on Cauchy's Theorem.

7. List as many methods as you can for evaluating contour integrals.

8. List some results which may be considered surprising when compared with their analogues in real analysis.

9. Give some examples of the use of (i) the Estimation Theorem, (ii) Morera's Theorem, (iii) Liouville's Theorem. (iv) Rouché's Theorem.

10. What results are used in the proof of Taylor's Theorem?

11. Compare and contrast the methods you might use for calculating the coefficients of Taylor series and Laurent series.

12. Why does complex analysis give us a better understanding of the convergence of the real series $1 - x^2 + x^4 - x^6 + \ldots$?

13. (i) Explain what it means to say that certain contour integrals are "path-independent". From what result does such a statement derive?

    (ii) Give an example of a contour integral which is path-independent, and an example of one which is path-dependent.

14. List some of the uses of the Residue Theorem.

15. Outline the development of the Laurent series for a punctured disc, starting with the Cauchy–Goursat Theorem.

Answers

Since the questions are open-ended, it follows that the answers we give may not coincide with yours. For example, it is possible that some of your answers for Question 1 are not included in our list.

1. The limit in question, $\lim_{z \to a} \dfrac{f(z) - f(a)}{z - a}$, is taken through complex values. Loosely speaking, we may say that f is differentiable at a if the limit exists no matter how z approaches the point a. Four of the consequences of this fact are as follows.
   (i) If a function is analytic on a region, then so is its derivative. (Loosely, if a function can be differentiated once, it can be differentiated any number of times.)
   (ii) If a function is differentiable near a point, it can be represented by a Taylor series.
   (iii) An analytic function is completely determined by its values at relatively few points (for example, the set of points on a contour).
   (iv) $x \longrightarrow |x|$ is differentiable everywhere except at 0; $z \longrightarrow |z|$ is differentiable nowhere. (See also the answer to Question 8.)

2. The concept of compactness is used in several places in this course, for example:
   (i) the proof of the Boundedness Theorem (*Unit 2*);
   (ii) the proof of the lemma to the Cauchy–Goursat Theorem (*Unit 5*);
   (iii) the definition of the *distance* between a point and a closed set (*Unit 2*).

3. The real function exp is one–one, but its complex analogue is not (since $\exp(z + 2n\pi i) = \exp z$, $n \in \mathbf{N}$). Because of this, we cannot define an inverse function unless we either restrict the domain or interpret the inverse function as a complete analytic function defined on a Riemann surface.

4. The trigonometric identities, such as $\sin^2 z + \cos^2 z = 1$ and $\sin 2z = 2 \sin z \cos z$, are preserved; also, the real and the complex sine functions are periodic with period 2π. However, it is *not* true that $|\sin z| \leq 1$ if $z \in \mathbf{C}$; in fact, we can solve equations such as $\sin z = 2$, which have no solutions in the real case.

5. The full story is given in *Unit 4*, but briefly:
   (i) we defined the integral of a complex-valued function;
   (ii) this enabled us to integrate along a smooth arc;
   (iii) we proved the Reparametrization Theorem;
   (iv) this enabled us to define the integral along a directed path;
   (v) by "stringing together" directed paths, we were able to define the integral along a contour.

6. Cauchy's Theorem is used to establish Cauchy's Formulas, which in turn lead to Taylor's Theorem, the Laurent series, and the Residue Theorem. Many other results are proved on the way, and we suggest that you refer to the structural diagram at the end of this booklet for these other results.

7. (i) Direct calculation, by parametrizing the contour (*Unit 4*).
   (ii) Using the Fundamental Theorem of Calculus, when the integrand is the derivative of some known function (*Unit 4*).
   (iii) Using Cauchy's Theorem (*Units 5* and *9*).
   (iv) Using Cauchy's Formulas, for integrals of the form
   $$\int_\Gamma \frac{f(z)}{(z-a)^n} dz \qquad (\textit{Units 5} \text{ and } \textit{9}).$$

(v) Using integration by parts and by substitution (*Unit 5*).

(vi) Using the calculus of residues (*Units 8, 9* and *10*).

8. In addition to those given in the answer to Question 1, we may include:

(i) Liouville's Theorem;

(ii) a uniformly convergent series of analytic functions may be differentiated term by term;

(iii) analytic functions are very restricted in their behaviour (for example, the Maximum Principle (*Unit 9*), the Principle of the Argument (*Unit 10*), the Uniqueness Theorem (*Unit 6*)).

9. (i) The Estimation Theorem is used in the proofs of the Cauchy–Goursat Theorem, Cauchy's Formulas, Taylor's Theorem, Liouville's Theorem, and in the evaluation of real improper integrals and the summation of series.

(ii) Morera's Theorem can be used to establish that a convergent power series defines an analytic function, and that a uniformly convergent sequence of analytic functions converges to an analytic function.

(iii) Liouville's Theorem can be used to give a neat proof of the Fundamental Theorem of Algebra.

(iv) Rouché's Theorem can also be used to prove the Fundamental Theorem of Algebra, and also to obtain information on the number and the location of the roots of an equation.

10. Cauchy's Formula (for a circle), Cauchy's Formula for derivatives (for a circle), the Boundedness Theorem, the Estimation Theorem. See also the structural diagram.

11. The Taylor coefficients are simply the derivatives of the function at a point, and this fact can be used to calculate them, although other methods depending on the properties of power series are often more useful (see Question 6.3 of this revision booklet). Calculating the coefficients of a Laurent series from their definition as an integral is almost always impractical, and so once again we often have to fall back on other methods involving the properties of power series.

12. The series $1 - z^2 + z^4 - \ldots$ converges to $\dfrac{1}{1+z^2}$; but the function $z \longrightarrow \dfrac{1}{1+z^2}$ is not analytic on the disc $|z| < r$, if $r > 1$, because of the poles at i and $-i$. It follows from Taylor's Theorem that the radius of convergence of the series must be 1. On the other hand, the real function $x \longrightarrow \dfrac{1}{1+x^2}$, $x \in \mathbf{R}$, appears (at first sight) to behave quite well, and there seems to be no apparent reason for the failure of the series to converge. In other words, the reason why the real series fails to converge if $|x| > 1$ is that the corresponding complex function has poles at i and $-i$.

13. (i) To say that $\int_\Gamma f$ is path-independent means that, if z_1 and z_2 are two points in a region R, and Γ is a contour in R joining them, then $\int_\Gamma f$ depends on f, z_1 and z_2, and not on the particular choice of Γ. Such statements derive essentially from Cauchy's Theorem (or from the Fundamental Theorem of Calculus, which is a special case of Cauchy's Theorem).

(ii) The integrals $\int_\Gamma e^z\,dz$ and $\int_\Gamma z\,dz$ are both path-independent, but $\int_\Gamma \dfrac{dz}{z}$ is path-dependent (since its value depends on which side of Γ the origin appears).

14. The Residue Theorem is used in:
    (i) the evaluation of various contour integrals, including the real trigonometric integrals;
    (ii) the evaluation of real improper integrals;
    (iii) the summation of series;
    (iv) the proof of the Principle of the Argument.

15. See the structural diagram.

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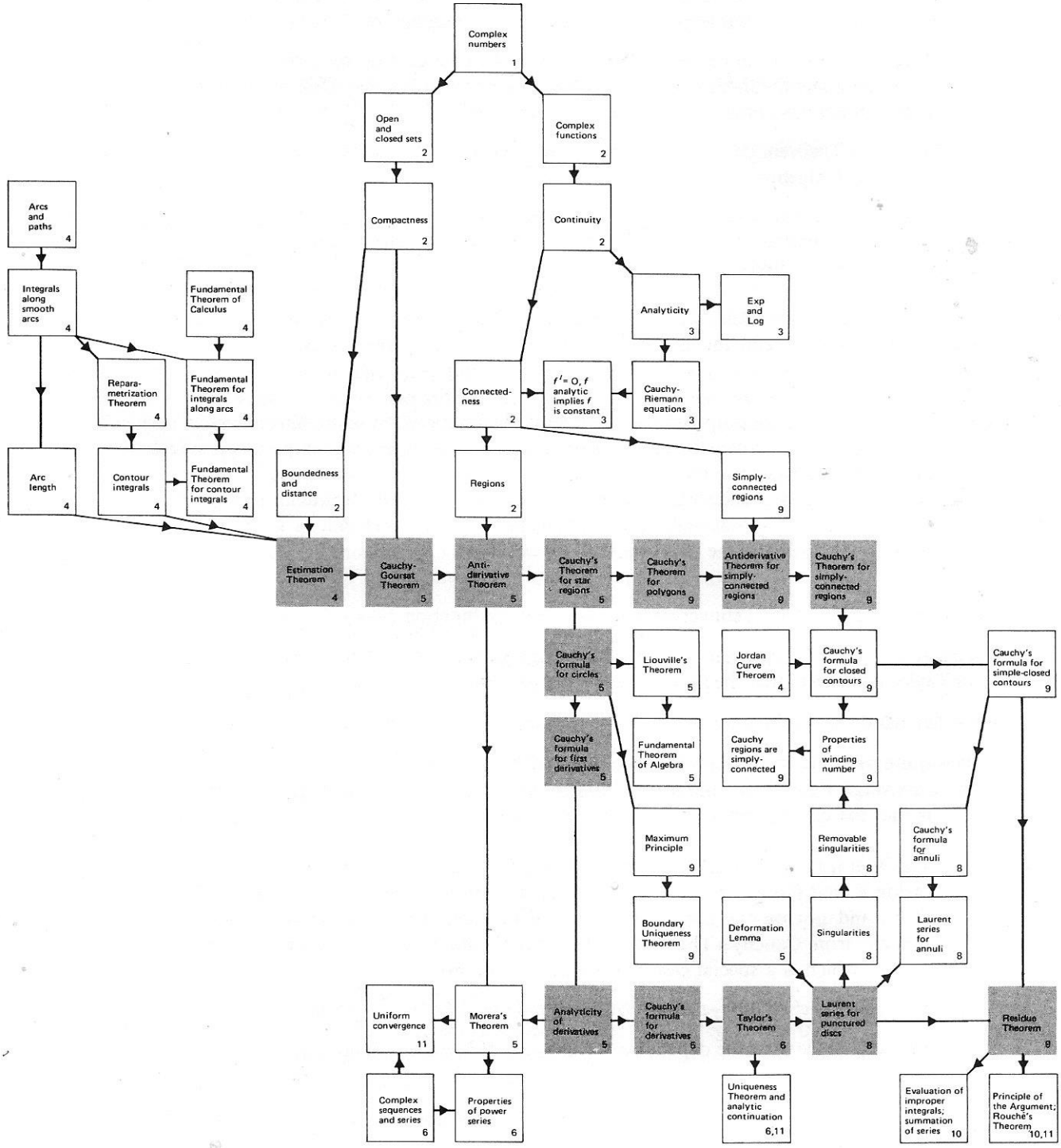
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